How much should the nation save?

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Notes to lectures 3 and 4

Capital accumulates according to

$$(1+\gamma)k_{t+1} = k_t + f(k_t) - c_t \qquad t = 0, 1, 2, ...$$

Golden rule approach (Phelps):

Assume balanced growth:

$$k_{t+1} = k_t = k \qquad c_t = c$$

$$\max_k c = f(k) - \gamma k$$

First order condition.

$$\frac{dc}{dk} = f'(k) - \gamma = 0$$

$$f'(k) = \gamma = n + g + ng$$

Utilitarian approach (Ramsey):

$$\max V = \sum_{t=0}^{\infty} \beta^t u(c_t A_t) L_t = u(c_0 A_0) L_0 + \beta u(c_1 A_1) L_1 + \beta^2 u(c_2 A_2) L_2 ...$$

given

$$c_t = k_t + f(k_t) - (1+\gamma)k_{t+1} \qquad t = 0, 1, 2, ...$$

$$k_0 = \bar{k_0} \qquad k_t \geq 0 \qquad \text{for } t = 0, 1, 2, ...$$

Max with respect to $c_0, c_1, c_2, ...$ and $k_1, k_2, ...$

$$0 < \beta < 1 \qquad \text{discount factor (subjective)}$$

$$\beta = \frac{1}{1+\rho} \qquad \rho = \text{discount rate, degree of impatience}$$

$$u(C) \qquad \text{period utility} \qquad u' > 0 \qquad u'' < 0$$

Simplification

$$A_t = 1 \qquad L_t = 1 \qquad t = 0, 1, 2, ...$$

$$\gamma = 0 \qquad \text{No natural growth.}$$

$$\max V = \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{given} \tag{1}$$

$$c_t = k_t + f(k_t) - k_{t+1} \qquad t = 0, 1, 2, ... \tag{2}$$

$$k_0 = \bar{k_0}, \qquad k_t \geq 0 \qquad t = 0, 1, 2, ... \tag{3}$$

Assume interior solution:
Insert for c_t from (2) in (1). Take derivatives w.r.t. k_{t+1}
k_{1+1} appear in $\sum$ in terms t and t+1:

$$V = ... + \beta^t u(f(k_t) + k_t - k_{t+1}) + \beta^{t+1} u(f(k_{t+1}) + k_{t+1} - k_{t+2}) + ...$$

1.o. cond:

$$\frac{\partial V}{\partial k_{t+1}} = \underbrace{-\beta^t u'(c_t)}_{(A)} + \underbrace{\beta^{t+1} u'(c_{t+1}) \underbrace{(f'(k_{t+1}) + 1)}_{(B)}}_{(C)} = 0 \qquad t = 0, 1, ... \tag{4}$$

(A) Utility loss from consuming less in period t.
(B) Increase in resources available in period t+1.
(C) Utility gain from increased consumption in period t+1.

1.o.cond. simplified (consumption Euler equation):

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \qquad t = 0, 1, 2, ... \tag{5}$$

Marginal utility today = Discount factor * Marginal utility tomorrow* Gross return on savings.

Alternative ways of writing 1.o.cond.

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + f'(k_{t+1})$$

MRS=MRT

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1 + f'(k_{t+1})}{1 + \rho}$$

Remember $\beta = 1/\rho$.

$$f'(k_{t+1}) = \rho \Rightarrow u'(c_{t+1}) = u'(c_t) \Rightarrow c_{t+1} = c_t$$

Marginal productivity of capital= Subjective discount rate constant implies constant consumption

$$f'(k_{t+1}) > \rho \Rightarrow u'(c_{t+1}) < u'(c_t) \Rightarrow c_{t+1} > c_t$$

$$f'(k_{t+1}) < \rho \Rightarrow u'(c_{t+1}) > u'(c_t) \Rightarrow c_{t+1} < c_t$$

Use that $u'' < 0$

Laws of motion

$$c_t = f(k_t) + k_t - k_{t+1} \tag{6}$$

$$u'(c_t) = \beta u'(c_{t+1})\beta(1 + f'(k_{t+1})) \tag{7}$$

$$k_0 = \bar{k_0}$$

System of two difference equations in two unknowns.
One initial condition, an infinity of solutions.
First order cond. not sufficient.
How to pin down the optimum? How much should we consume now? Look forward!
(Closed form solutions not available.)

Steady state

Definition of steady state:

$$k_{t+1} = k_t = k^*$$

$$c_{t+1} = c_t = c^*$$

From (8) and (9). S.s. conditions:

$$c^* = f(k^*) \tag{8}$$

$$\beta(1 + f'(k^*)) = 1 \tag{9}$$

(6) determines c_*, (7) determines k_*

Golden rule:	$f'(k^{**})$	$=$	0	(since n=g=0)
Ramsey rule:	$f'(k^*)$	$=$	ρ	$k^* < k^{**}$

Impatience means less saving and a lower steady state capital stock than the golden rule.

Dynamics

Need to determine starting point for c (or the *level* of the consumption path).

In figure 1 the curve $\Delta k = 0$ shows the combinations of k and c that yields a constant capital intensity. In an economy without natural growth the curve is the same as the production function. If consumption is above the curve, the capital intensity will be declining (arrow pointing left). Consumption below the curve means that the capital intensity is increasing.

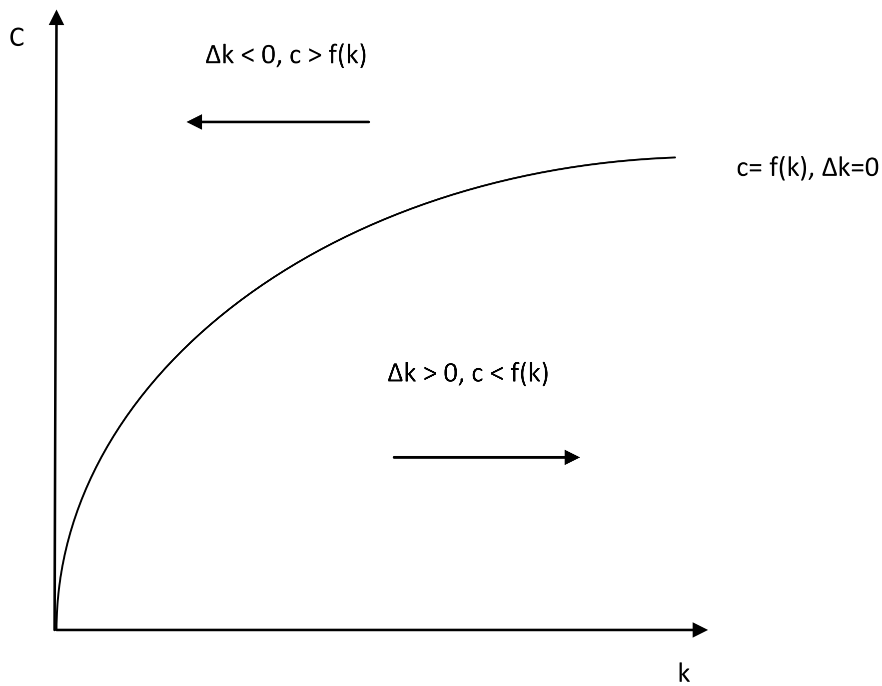

Figure 1: The movement of capital

Typo at the top of figure 2. Expression should be $f'(k) = \rho$.

The social planner chooses a constant consumption path when $f'(k) = \rho$, which happens for $k = k_*$. To the right of k_* return on capital is lower. It does not compensate fully for the planners impatience and thet social planner therefore chooses a declining consumption path (arrow pointing down). To the left of k_* the return to capital is higher than ρ. It is more than enough to compensate for impatience and the social planner will choose an increasing consumption path.

> Figure 2 is not 100 per cent accurate. On the axises of the graph should be k_t and c_t. However, from the first-order condition $c_{t+1} = c_t$ when $f'(k_{t+1}) = \rho$, not when $f'(k_t) = \rho$. Strictly speaking $c_{t+1} = c_t$ requires $c_t = k_t + f(k_t) - k_*$, which means that the vertical line in figure 2 should be replaced by an upward sloping curve, the slope being $d_c/dk = 1 + f'(k)$. However, an

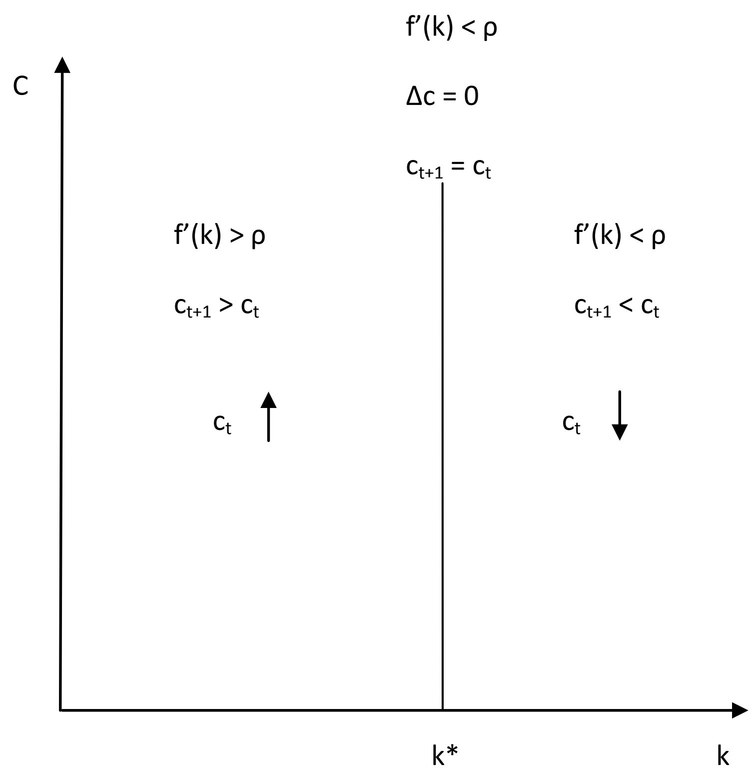

Figure 2: The movement of consumption

argument can be made that when the period length goes to zero, the slope of the curve goes to infinity. Hence, when the period is short we make no significant error by drawing the curve vertical. Another way of explaining this is that when the period is short k_{t+1} is close fo k_t. No conclusions are changed if we make the curve slope upwards. If this comment make you worry, then suppress it.

Figure 3 combines figures 1 and 2. The initial value for k, k_0, is given. The initial value for c remains to be determined. Try different starting points:

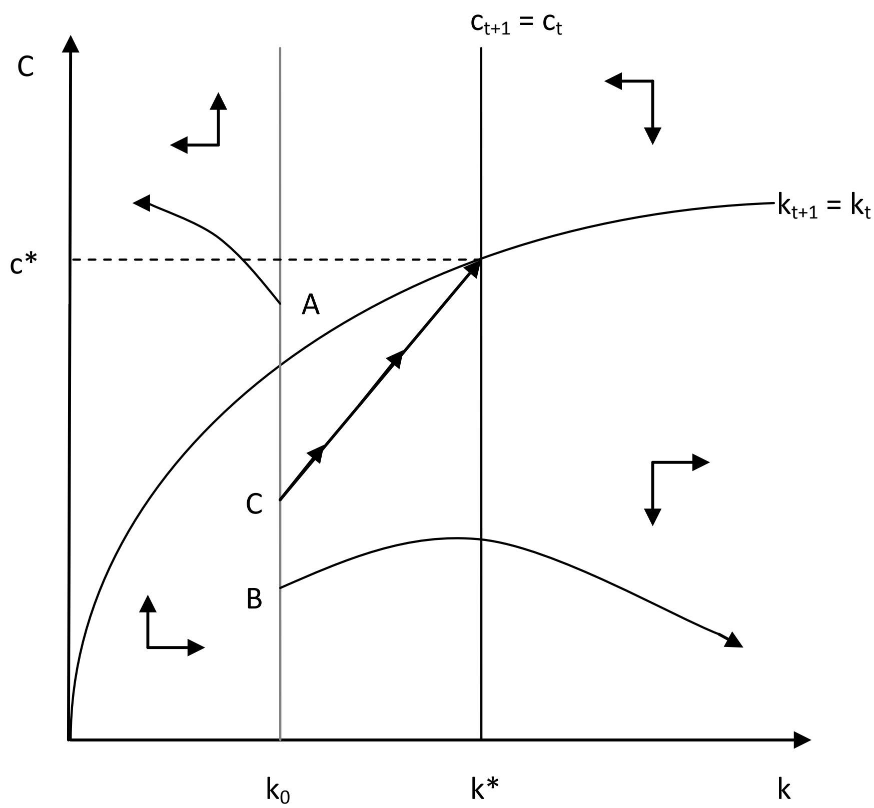

Figure 3: The joint movement of consumption and capital

A Consumption exceeds output, and hence the capital stock is declining. First-order conditions indicate an increasing consumption path. Capital stock ends up negative. Path is unsustainable, violates the resource constraint.

B Consumption is lower than output, and hence the capital stock is increasing. After a while consumption enters the region where the first-order condition says that consumption should decline. Investing more and more and consuming less and less forever is clearly inefficient. Resources are not Fully utilized.

C The path ends in the stationary point. It is sustainable (does not violate the resource constraint). It has higher consumption than all paths starting below C. All paths starting above C are unsustainable. Hence the path from C is efficient and solves the original maximization problem. *Saddle path.*